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Short Communication

Towards improved evaluation of large amplitude free-vibration behaviour of uniform beams using multi-term admissible functions

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1. Introduction

A study of large amplitude free vibrations of simply supported beams, with axially immovable ends, is presented in the classic work of Woinowsky-Krieger [1], wherein an elegant solution is obtained in terms of elliptic integrals. Subsequently, this interesting problem is solved using the versatile finite element method, with some assumptions and approximation [2,3]. The main advantage of the finite element method is its applicability to beams with different boundary conditions. A continuum solution for the same problem was presented by Srinivasan [4] through the Ritz–Galerkin method with an assumed space–time distribution for the lateral displacement. Singh et al. [5] proposed a refined finite element formulation, which gives very accurate results for the aforementioned problem. A detailed presentation of large amplitude vibration of beams can be seen in Ref. [6].

In general, trigonometric functions are used to represent lateral displacement distributions in the continuum solutions mentioned above. While these functions can be chosen to represent the exact behaviour of pinned–pinned beams, their choice can only represent approximately the behaviour of beams with other end conditions. Further, in all the earlier investigations, wherever applicable, only one-term trigonometric functions are used and their accuracy cannot be improved by including additional terms. For example, for a typical case of a clamped–clamped

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beam, a second term combined with the one-term solution cannot be obtained to represent the deformation of the beam in the first vibration mode.

However, polynomial functions with multiple terms can be used with advantage satisfying different types of geometric boundary conditions of the beams. Multiple term polynomial functions are derived and employed in this paper to study the large amplitude free vibrations of clamped–clamped and pinned–clamped beams using the conservation of total energy principle developed earlier by the authors [7]. It may be noted that as a simple trigonometric function gives exact solution [1] for pinned–pinned beam, this case is not considered in the present study. The ratios of nonlinear to linear frequencies for different maximum amplitudes obtained presently are compared with accurate finite element solutions of Singh et al. [5].

In the following sections, the formulation for large amplitude free vibration of uniform beams involving the multi-term approximations to the lateral displacement is presented along with numerical results and discussion.

2. Formulation

For any vibrating system, neglecting damping, the total energy at any given instant of time is constant and is written as

$$T + U + W = \text{Constant}, \tag{1}$$

where T is the kinetic energy, U the strain energy and W the potential energy arising due to the axial tension 'P' developed due to large amplitudes, but with small strains.

For the case of beams with immovable ends

$$T = \frac{m}{2}\int_0^L (\dot{w})^2\,\mathrm{d}x, \tag{2}$$

$$U = \frac{EI}{2}\int_0^L (w'')^2\,\mathrm{d}x \tag{3}$$

and

$$W = \frac{P}{2}\int_0^L \tfrac{1}{2}(w')^2\,\mathrm{d}x, \tag{4}$$

where m is the mass per unit length, w the lateral displacement, L the length of the beam, E the Young's modulus, I the area moment of inertia, x the axial coordinate, $(\dot{})$ denotes differentiation with respect to time 't' and $()'$ denotes differentiation with respect to 'x'.

From Eqs. (1)–(4), we have

$$m\int_0^L (\dot{w})^2\,\mathrm{d}x + EI\int_0^L (w'')^2\,\mathrm{d}x + P\int_0^L \tfrac{1}{2}(w')^2\,\mathrm{d}x = \text{Constant}. \tag{5}$$

The axial tensile load 'P' for a beam with immovable ends can be obtained as follows:

Assuming that one end of the beam is movable, the shortening of the length ΔL_1 is given by

$$\Delta L_1 = \frac{1}{2}\int_0^L (w')^2\,\mathrm{d}x. \tag{6}$$

If the axial tensile load 'P' is acting on the beam, the axial elongation 'ΔL_2' due to 'P' is

$$\Delta L_2 = \frac{PL}{AE}, \tag{7}$$

where A is the area of cross-section of the beam. For a beam with immovable ends,

$$\Delta L_1 = \Delta L_2. \tag{8}$$

From Eq. (8), we get

$$P = \frac{AE}{2L}\int_0^L (w')^2\,\mathrm{d}x. \tag{9}$$

Noting that

$$I = Ar^2, \tag{10}$$

where r is the radius of gyration, Eq. (9) can be written as

$$P = \frac{EI}{2Lr^2}\int_0^L (w')^2\,\mathrm{d}x. \tag{11}$$

The lateral displacement, $w(x,t)$ is expressed as

$$w(x,t) = W.\delta, \tag{12}$$

where W is the nondimensional spatial distribution of lateral displacement at any time, t, normalised in such a way that the maximum value is unity and δ represents the time-dependent lateral displacement at any x, the maximum being δ_m at the point max $x_{\max}$, where the maximum lateral displacement occurs.

Substituting Eq. (12) into Eq. (5), we get

$$\begin{aligned}\dot{\delta}^2 \int_0^1 W^2\,\mathrm{d}\xi &+ \frac{EI}{m}\,\delta^2 \int_0^1 (W'')^2\,\mathrm{d}\xi \\ &+ \left\{\frac{EI}{2mLr^2}\,\delta^2 \int_0^1 (W')^2\,\mathrm{d}\xi\right\}\left\{\tfrac{1}{2}\delta^2 \int_0^1 (W')^2\,\mathrm{d}\xi\right\} = \text{Constant}.\end{aligned} \tag{13}$$

Substituting the functional forms of W (see Appendix A) for the clamped–clamped and pinned–clamped beams and after simplification the temporal equation is obtained as

$$\dot{\delta}^2 + \alpha_1\delta^2 + \alpha_2\delta^4 = \text{Constant}(H). \tag{14}$$

For the clamped–clamped beam

$$\alpha_1 = 504.00\,\frac{EI}{mL^4} \tag{15}$$

and

$$\alpha_2 = 14.6286 \frac{EI}{mL^4 r^2} \tag{16}$$

for a one-term solution and

$$\alpha_1 = 500.5847 \frac{EI}{mL^4} \tag{17}$$

and

$$\alpha_2 = 15.0196 \frac{EI}{mL^4 r^2} \tag{18}$$

for a two-term solution.

For the pinned–clamped beam, α_1 and α_2 are

$$\alpha_1 = 238.7368 \frac{EI}{mL^4} \tag{19}$$

and

$$\alpha_2 = 14.4222 \frac{EI}{mL^4 r^2} \tag{20}$$

for a one-term solution and

$$\alpha_1 = 238.4820 \frac{EI}{mL^4} \tag{21}$$

and

$$\alpha_2 = 14.6075 \frac{EI}{mL^4 r^2} \tag{22}$$

for a two-term solution.

3. Linear free vibration

Neglecting α_2 in Eq. (14), we get the linear frequency of the beam. The constant 'H' for the linear vibration case can be easily obtained, by using the condition

$$\dot{\delta} = 0 \quad \text{at } \delta = \delta_m, \tag{23}$$

where δ_m is the maximum amplitude of vibration, as

$$H = \alpha_1 \delta_m. \tag{24}$$

Then from Eq. (14)

$$\dot{\delta} = \frac{\mathrm{d}\delta}{\mathrm{d}t} = [\alpha_1(\delta_m^2 - \delta^2)]^{1/2} \tag{25}$$

or

$$\mathrm{d}t = \frac{\mathrm{d}\delta}{[\alpha_1(\delta_m - \delta^2)]^{1/2}}. \tag{26}$$

Integrating Eq. (26), we get the linear time period (T_L) as

$$T_L = \frac{2\pi}{\omega} = 4\int_0^{\delta_m} \frac{\mathrm{d}\delta}{[\alpha_1(\delta_m - \delta^2)]^{1/2}}, \tag{27}$$

where ω_L is the radian frequency of the beam in the linear range.

Substituting

$$\delta = \delta_m \sin\theta \tag{28}$$

into Eq. (27) and integrating the right-hand side, the time period T_L is obtained as

$$T_L = \frac{2\pi}{\omega_L} = \frac{2\pi}{\alpha_1^{1/2}}. \tag{29}$$

Eq. (29) gives the radian frequency of the beam in the linear range.

4. Large amplitude free vibration

When the beam is undergoing large amplitude free vibration, the coefficient α_2 in Eq. (14) has to be taken into account and the constant H in this case is

$$H = \alpha_1\delta_m^2 + \alpha_2\delta_m^4. \tag{30}$$

Then, Eq. (25) becomes

$$\dot{\delta} = [\alpha_1(\delta_m^2 - \delta^2) + \alpha_2(\delta_m^4 - \delta^4)]^{1/2}. \tag{31}$$

The nonlinear period T_{NL} after integration is

$$T_{NL} = \frac{2\pi}{\omega_{NL}} = 4\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\{\alpha_1[1 + \frac{\alpha_2}{\alpha_1}(1 + \sin^2\theta)\delta_m^2]\}^{1/2}}, \tag{32}$$

where ω_{NL} is the radian frequency of the nonlinear system for a given maximum amplitude of vibration δ_m.

The right-hand side of Eq. (32) can be integrated using a suitable numerical integration scheme, to obtain ω_{NL}. In the present study, Simpson's numerical integration rule is used to numerically integrate the right-hand side of Eq. (32).

5. Numerical results and discussion

Using the formulation presented in the previous sections, the ratios of nonlinear frequency ω_{NL} to the linear frequencies ω_L are obtained for several values of the maximum amplitude δ_m for the uniform clamped–clamped and pinned–clamped beams.

Table 1
Ratios of ω_{NL} to ω_L for a clamped–clamped beam

$\frac{\delta_m}{r}$	Present solution		Singh et al. [5]	Trignometric admissible function [8]
	One term	Two term		
0.0	1.0	1.0	1.0	1.0
0.2	1.0009	1.0009	1.0009	1.0009
0.4	1.0035	1.0036	1.0036	1.0037
0.6	1.0078	1.0081	1.0080	1.0084
0.8	1.0138	1.0143	1.0142	1.0149
1.0	1.0215	1.0222	1.0221	1.0231
2.0	1.0831	1.0858	1.0854	1.0892
3.0	1.1778	1.1833	1.1825	1.1902
4.0	1.2979	1.3067	1.3055	1.3178
5.0	1.4369	1.4492	1.4474	1.4647
λ_L	504.0	500.58	500.62	519.52

Table 1 gives the ω_{NL}/ω_L ratios of the clamped–clamped beam obtained with one- and two-term polynomial admissible functions for various values of amplitude parameter δ_m/r. The present results are compared with the accurate results of Singh et al. [5], obtained by the versatile finite element method. It can be seen from this table that the present two-term solution is more accurate than the one-term solution and match very well with those given in Ref. [5]. The solution obtained from the widely used trigonometric admissible function for clamped–clamped beam [8] shows a larger deviation from the finite element solution, both in the linear frequency parameter λ_L $(= m\omega_L^2 L^4/EI)$ value and the nonlinear to linear frequency ratio, ω_{NL}/ω_L. It may be noted as discussed in Refs. [9,10] that the effect of magnitude of axial force, P on the modal shapes is negligible and the present evaluation provides improved solution.

The ratios of ω_{NL}/ω_L for the pinned–clamped beam for various δ_m/r obtained through the present formulation are given for the one- and two-term solutions along with the results of Singh et al. [5] in Table 2. It can be noted from this table that the present two-term solution matches well with the accurate finite element solution [5].

6. Conclusions

Usefulness of the admissible polynomial functions with multiple terms in evaluating the nonlinear (large amplitude) free-vibration behaviour of beams, with axially immovable ends, is discussed in this note. The formulation is based on the principle of conservation of the total energy of the vibrating system. The temporal equation obtained directly from this principle can be integrated by any standard numerical integration scheme to obtain the ratios of the nonlinear to linear frequencies for various maximum amplitude ratios.

One- and two-term solutions are obtained for the uniform clamped–clamped beam and pinned–clamped beam. The present two-term solutions show a very good agreement with the accurate finite element solutions available in the literature.

Table 2
Ratios of ω_{NL} to ω_L for pinned–clamped beam

$\frac{\delta_m}{r}$	Present solution		Singh et al. [5]
	One term	Two term	
0.0	1.0	1.0	1.0
0.2	1.0018	1.0018	1.0019
0.4	1.0072	1.0073	1.0077
0.6	1.0162	1.0164	1.0172
0.8	1.0285	1.0289	1.0304
1.0	1.0442	1.0448	1.0471
2.0	1.1665	1.1676	1.1758
3.0	1.3416	1.3458	1.3615
4.0	1.5535	1.5599	1.5838
5.0	1.7885	1.7970	1.8293
λ_L	238.74	238.48	237.73

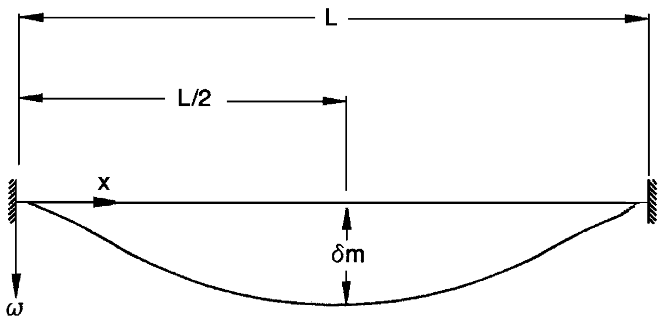

Fig. 1. A clamped–clamped beam.

Appendix A

The polynomial admissible functions for the clamped–clamped and pinned–clamped beams are obtained using the boundary conditions and the symmetry conditions.

A.1. Clamped–clamped beam

The boundary conditions for a clamped–clamped beam at $\xi(= x/L) = 0$ and 1 (Fig. 1) are

$$W(0) = W'(0) = W(1) = W'(1) = 0. \tag{A.1}$$

The symmetry condition is

$$W'\left(\tfrac{1}{2}\right) = 0. \tag{A.2}$$

Assuming a sixth degree polynomial

$$W(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5 + a_6\xi^6 \tag{A.3}$$

and using the boundary and symmetry conditions as Eqs. (A.1) and (A.2), $W(\xi)$ can be obtained to be

$$W(\xi) = (\xi^2 - 2\xi^3 + \xi^4) - b(\xi^3 - 3\xi^4 + 3\xi^5 - \xi^6). \tag{A.4}$$

Further, $W(\xi)$ has to be properly normalised so that the maximum displacement becomes unity.

Eq. (A.4) represents a two-term polynomial admissible function for the clamped–clamped beam with both the terms satisfying the boundary conditions and symmetry conditions. The solution can be obtained by using the first term alone (one-term solution) or both terms together (two-term solution). It is to be noted here that W has to be normalised in such a way that its value is unity at $\xi = \frac{1}{2}$. Eventhough this condition can be achieved in a simple way for one-term solution, the eigenvector corresponding to the linear frequency ω_L has to be evaluated in the two-term solution and then W has to be accordingly normalised to satisfy the aforementioned condition.

A.2. Pinned–clamped beam

For the pinned–clamped beam (Fig. 2), the boundary conditions are

$$W(0) = W''(0) = W(1) = W'(1) = 0 \tag{A.5}$$

and there are no symmetry conditions.

Assuming a quintic polynomial, the two terms of the admissible function for this beam are obtained as

$$W(\xi) = (\xi - 3\xi^3 + 2\xi^4) + b(\xi^3 - 2\xi^4 + \xi^5). \tag{A.6}$$

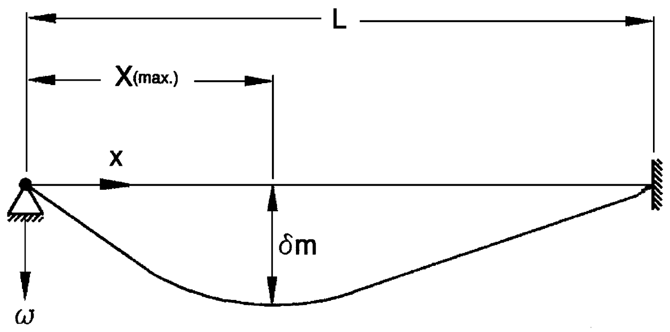

Fig. 2. A pinned–clamped beam.

Again as in the case of the clamped–clamped beam, W has to be normalised so as to obtain a unit displacement at $\xi = \xi_{\max}$. $\xi_{\max}$ can be calculated by obtaining the lowest root of

$$W'(\xi) = 0. \tag{A.7}$$

For a one-term approximation $\xi_{\max} = 0.41654$ and its value for the two-term solution is 0.42122.

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